

Eigenvalue repulsion estimates and some applications for the one-dimensional Anderson model

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Abstract. We show that the spacing between eigenvalues of the discrete 1D Hamiltonian with *arbitrary potentials* which are bounded, and with Dirichlet or Neumann Boundary Conditions is bounded away from zero. We prove an explicit lower bound, given by Ce^{-bN} , where N is the lattice size, and C and b are some finite constants. In particular, the spectra of such Hamiltonians have no degenerate eigenvalues. As applications we show that to leading order in the coupling, the solution of a nonlinearly perturbed Anderson model in one-dimension (on the lattice) remains exponentially localized, in probability and average sense for initial conditions given by a unique eigenfunction of the linear problem. We also bound the derivative of the eigenfunctions of the linear Anderson model with respect to a potential change.

1. Introduction

We consider the one dimensional Anderson model on the lattice, Λ ,

$$H_\omega^\Lambda u_n(x) = u_n(x+1) + u_n(x-1) + \varepsilon_x u_n(x) = E_n u_n(x), \quad (1.1)$$

with $x, n \in \mathbb{Z}$, $\omega = \{\varepsilon_x\}$ is the realization of the potential, H_ω^Λ is the Hamiltonian on the domain Λ , with eigenfunctions $\{u_n(x)\} \in L^2(\Lambda)$ and eigenvalues E_n . We will also denote by $N \equiv |\Lambda|$ the size of the domain. Furthermore, H_ω^Λ satisfies some boundary conditions to be specified later, which include both the Dirichlet and Neumann boundary conditions. Our first result applies to *arbitrary* uniformly bounded potential,

$$\sup_{x \in \Lambda} |\varepsilon_x| \equiv W < \infty. \quad (1.2)$$

We will show in the next section that the minimal distance between the eigenvalues of H_ω^Λ is bounded below by a constant of order e^{-bN} for every ω and as long as $W < \infty$ and the boundary conditions defining H_ω^Λ are of the allowed class. Note, that this result holds for *all* bounded potentials. Our proof, while not necessarily the simplest one, is instructive and may be of more general interest.

Then, in the next section we show two applications, motivated by the study of Anderson localization problem, both linear and nonlinear. In particular, in [11, 12] we have shown that for the nonlinearly perturbed Anderson model,

$$i\partial_t \psi = H_\omega^\Lambda \psi + \beta |\psi|^2 \psi, \quad (1.3)$$

with the initial condition of $\psi(x, 0) = u_0(x)$, the first order nonlinear correction to the solution is given by,

$$\psi^{(1)}(x, t) = \beta \sum_n c_n^{(1)}(t) u_n(x) e^{-iE_n t}, \quad (1.4)$$

with

$$c_n^{(1)}(t) = \frac{V_n^{000}}{E_n - E_0} (1 - e^{i(E_n - E_0)t}). \quad (1.5)$$

Higher order corrections involve products of $c_n^{(1)}$ and other combinations of energies. Relevant estimates were recently proven for such combinations in [4]. Note, that since H_ω^Λ depends on the realization of the potential, ω , so is $u_0(x)$. We will show here that on average, the fractional power of the solution of (1.3) to the first order in β , remains exponentially bounded for all times, we also show that the ordinary average is exponentially bounded at least for times which are exponential in N .

In the second application we control the averages of fractional powers of the derivative of the eigenfunctions of H_ω^Λ with respect to some ε_x , and show that they are exponentially small in the distance between x and the localization center of the eigenfunction. We also bound the averages by some power of the volume of the system and interpolate between the fractional and ordinary averages.

The proof of the eigenvalue repulsion is based on the transfer matrix representation of the solutions of the one dimensional problem, and study the dependence of the

eigenfunctions on the energy [8]. By studying the properties of the matrices as transformations of the Hyperbolic space, in terms of the complex energy as a parameter, the presence and absence of continuous spectrum for classes of Random Schrödinger operators on graphs can be naturally analyzed [6]. This is close to our approach. We then show, that the condition for the energy parameter to have a value that corresponds to an eigenvalue, requires a path to return to the starting point, in some sense. Next, we prove monotonicity of a rotation number/angle associated with the path, as a function of the energy parameter. Monotonicity with respect to the energy parameter, is also used in the hyperbolic space representation; there, it appears as a basic property of the Mobius transformation [5]. By bounding the rate of rotation from above, as a function of the energy parameter, a minimal distance between the eigenvalues follows.

The applications mentioned above, use the exponentially small minimal distance between the eigenvalues in a crucial way. We decompose dyadically the space of potentials, ω , to subsets where the minimal distance between eigenvalues is in a dyadic interval, $I_m \in [2^{-m-1}, 2^{-m}]$. Then, the sum over m is bounded up to, $m \leq \bar{b}N$, due to the eigenvalue repulsion. We estimate each term by a combination of two probabilistic estimates: first, the Minami estimate for the probability to find at least two eigenvalues in an interval I , [14]

$$\Pr \left(\text{Tr} P_{H_\omega^\Lambda}^{(\Lambda)}(I) \geq 2 \right) \leq (\pi \|\rho\|_\infty I N)^2, \quad (1.6)$$

where $\|\rho\|_\infty$ is the supremum of the density of states, I is some energy interval while $P_{H_\omega^\Lambda}^{(\Lambda)}(I)$ is the spectral projection on that interval and H_ω^Λ is the Hamiltonian corresponding to a one-dimensional Anderson problem with Dirichlet boundary conditions on a domain Λ . The second bound we use is the fractional moment bound of Aizenman [1] (see also related bounds in [2, 7]),

$$\left\langle \sum_n |u_n(x) u_n(y)| \right\rangle \leq D e^{-\mu|x-y|}, \quad (1.7)$$

where $\mu > 0$, and $D > 0$ are some constants.

2. Lower bound on level spacings

2.1. Main Result

The Main Result is the following Theorem:

Theorem 1. (*eigenvalue repulsion*) *Given the tight binding model:*

$$H u_n = u_{n-1} + \varepsilon_n u_n + u_{n+1}; \quad 1 \leq n \leq N$$

with Dirichlet boundary conditions ($u_0 = 0, u_{N+1} = 0$) or Neumann boundary conditions ($u_0 = u_1, u_{N+1} = u_N$) and with $0 \leq \varepsilon_n \leq W < \infty$ for all n , there exists a constant $0 < \eta(W) < 1$ such that: $|E_i - E_j| \geq \frac{\pi(\eta^{-1}(W)-1)}{\eta^{-N}(W)-1}$ are for all $i \neq j$ and eigenvalues E_i, E_j (in this Section as well as in Appendix A and Appendix B lattice sites are denoted by n).

2.2. Setup

For simplicity, we first prove the main Theorem for Dirichlet b.c., and then describe the modifications needed for Neumann case in a separate subsection. Obviously, E is an eigenvalue of H , if and only if, there exists a non-trivial vector $\vec{u} = \{u_n\}_{n=0}^{N+1}$ so that $H\vec{u} = E\vec{u}$ and:

$$u_0 = 0 \quad (2.1)$$

$$u_{N+1} = 0. \quad (2.2)$$

Since $u_0 = u_1 = 0$ implies $\vec{u} \equiv 0$, we can set (without loss of generality):

$$u_1 = 1. \quad (2.3)$$

For arbitrary E , given (2.1), (2.3) we can calculate all the components u_n of \vec{u} by recursive formula:

$$u_{n+1}(E) = (E - \varepsilon_n)u_n(E) - u_{n-1}(E). \quad (2.4)$$

E is an eigenvalue of H iff (2.2) holds.

Definition 2. Let $\alpha_n(E)$ be the angle between the 2D vector $(u_{n-1}(E), u_n(E))$, and the positive direction of abscissa at the corresponding Cartesian plane. $\varphi_n(E)$ is said to be a *version of angular ratio* between $u_{n-1}(E)$ and $u_n(E)$, iff $\exists m \in \mathbb{Z}$, so that $\alpha_n(E) + 2m\pi = \varphi_n(E)$.

By the definition 2, it holds for $k \in \mathbb{Z}$:

$$\varphi_{N+1} = k\pi \Leftrightarrow u_{N+1} = 0 \Leftrightarrow E - \text{eigenvalue}. \quad (2.5)$$

Also:

$$(i) \text{ If } u_{n-1} \neq 0 \text{ then: } \tan(\varphi_n) := \frac{u_n}{u_{n-1}},$$

$$(ii) \text{ If } u_n \neq 0 \text{ then: } \cot(\varphi_n) := \frac{u_{n-1}}{u_n}.$$

By the recursive formula (2.4), we have:

$$\varphi_{n+1}(E) = \begin{cases} \arctan(E - \varepsilon_n - \cot \varphi_n(E)) + (k + 2m)\pi & k\pi < \varphi_n(E) < (k + 1)\pi \\ (k - \frac{1}{2})\pi + 2m\pi & \varphi_n(E) = k\pi \end{cases} \quad (2.6)$$

where $k, m \in \mathbb{Z}$ and $\arctan : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$.

The upper row of (2.6) can be obtained by dividing (2.4) by u_n , identifying $\cot(\varphi_n) := \frac{u_{n-1}}{u_n}$, taking *arctan* of both sides, and then adding arbitrary integer number of full rotations m . (In what follows $m = 0$ will be chosen so that φ_n are differentiable as functions of E .)

The lower row of (2.6), refers to the case when $\varphi_n(E) = k\pi \Leftrightarrow u_n = 0 \Rightarrow u_{n+1} = -u_{n-1}$. Then, it is obvious by considering the directions of 2D vectors $(u_{n-1}, 0)$ and $(0, -u_{n+1})$. Once again, arbitrary integer number of full rotations is added.

By (2.6), the sequence $\{\varphi_n\}$ is well defined up to addition of $2m\pi$, $m \in \mathbb{N}$. Therefore, any version of (2.6) can be used, to verify if the condition (2.5) holds for an energy E . In particular, we can set $m \equiv 0$ and choose $\{\varphi_n\}_{n=1}^{N+1}$ to be:

$$\begin{aligned} \varphi_1(E) &\equiv \frac{\pi}{2} \\ \varphi_{n+1}(E) &= \begin{cases} \arctan(E - \varepsilon_n - \cot \varphi_n(E)) + k\pi & k\pi < \varphi_n(E) < (k+1)\pi \\ \varphi_{n+1}(E) = (k - \frac{1}{2})\pi & \varphi_n(E) = k\pi \end{cases} \end{aligned} \quad (2.7)$$

The version (2.7) of (2.6) is especially convenient for our further use, because $\varphi_{N+1}(E)$ turns to be a continuously differentiable function of E (see Proposition 3).

The angle variable φ_n is known as the Prüffer angle [10]‡.

2.3. Proof

2.3.1. Proof for Dirichlet boundary conditions Eigenvalues satisfy (2.5) as explained in the setup section. We will show that $\varphi_{N+1}(E)$ rotates monotonously counterclockwise (Proposition 3), and that there is no degeneracy (Proposition 4).

We then show that the rotation speed φ'_{N+1} , is bounded from above (Proposition 5). But $\varphi_{N+1}(E)$ must change by angle of π between every pair of eigen-energies, (see (2.5)), and its rotation speed is bounded from above, therefore, the spacing between eigenvalues is bounded from below; that is: $|E_{i+1} - E_i| \geq \frac{\pi}{\varphi'_{N+1, \max}}$.

Proposition 3. $\varphi_{N+1}(E)$ is a continuously differentiable and a strictly increasing function of E .

Proof. $\varphi_2(E) = \arctan(E - \varepsilon_1)$ is continuously differentiable and $\varphi'_2(E) > 0$. Next we use induction in n :

If $\varphi_n \neq k\pi$, then $\varphi_{n+1}(E)$ is continuously differentiable and increasing, according to the definition, since $\arctan(\cdot)$ is a strictly increasing differentiable function of its argument. The argument of $\arctan(\cdot)$: $E - \cot(\varphi_n)$ is strictly increasing and continuously differentiable (by induction assumption on φ_n starting from φ_2).

If $\varphi_n = k\pi$, and continuously differentiable and increasing (with respect to E), then: $\varphi_{n+1}(E)$ is continuous, because the single side limits (2.8) and (2.9) are equal to each other as follows from the definition (2.7):

$$\begin{aligned} \lim_{\varphi_n \rightarrow k\pi^-} \varphi_{n+1}(E) &= \pi(k-1) + \lim_{\varphi_n \rightarrow k\pi^-} \arctan(-\cot(\varphi_n)) = \\ &= \pi(k-1) + \arctan(-(-\infty)) = k\pi - \frac{\pi}{2} \end{aligned} \quad (2.8)$$

$$\begin{aligned} \lim_{\varphi_n \rightarrow k\pi^+} \varphi_{n+1}(E) &= k\pi + \lim_{\varphi_n \rightarrow k\pi^+} \arctan(-\cot(\varphi_n)) = \\ &= k\pi + \arctan(-(+\infty)) = k\pi - \frac{\pi}{2} \end{aligned} \quad (2.9)$$

‡ We thank Michael Aizenman for bringing this to our attention

left and right “single-sided” derivatives of $\varphi_{n+1}(E)$ exist and equal. That is because, for any $\varphi_n \neq k\pi$ point, it holds:

$$\varphi'_{n+1}(E) = \frac{d\varphi_{n+1}(E)}{dE} = \frac{1 + \frac{\varphi'_n}{\sin^2 \varphi_n}}{1 + (E - \varepsilon_n - \cot \varphi_n)^2} \quad (2.10)$$

Taking the single side limits at $\varphi_n = k\pi$, and recalling that φ_{n+1} is continuous, one gets:

$$\begin{aligned} \lim_{\varphi_n \rightarrow k\pi^+} \varphi'_{n+1}(E) &= \lim_{\varphi_n \rightarrow k\pi^-} \varphi'_{n+1}(E) = \varphi'_n(E) \\ \Rightarrow \varphi'_{n+1}(E) &= \varphi'_n(E). \end{aligned} \quad (2.11)$$

Hence derivative exists, and (by induction) is positive as required. \square

Proposition 4. *The spectrum of H is simple.*

General proof of the simplicity of spectrum is given in theorem 7. Simplicity of spectrum can also be shown using “ φ ” (Pruffer angle [10]) formalism used here:

Proof. To ensure that no degeneracy occurs (that is $|E_i - E_j| \neq 0$), we need to show that solutions of $\varphi_{N+1}(E) = k\pi$ are simple. It is sufficient to show that $\varphi'_{N+1}(E) \neq 0$.

We saw that $\varphi'_2(E) > 0$ consequently:

For $\varphi_n(E) \neq 0$, we have that: $\varphi'_{n+1}(E) = \frac{1 + \frac{\varphi'_n}{\sin^2 \varphi_n}}{1 + (E - \varepsilon_n - \cot \varphi_n)^2} > 0$ (by induction assumption).

For $\varphi_n(E) = 0$, $\varphi'_{n+1}(E) = \varphi'_n(E) > 0$ (by induction assumption).

Therefore $\varphi'_{N+1}(E) > 0$ and there is no degeneracy.

Another way to ensure absence of degenerate eigenvalues of H is by successive use of (2.7), and considering limits of $\varphi_{N+1}(E)$ at $E \rightarrow \pm\infty$:

$$\begin{aligned} \lim_{E \rightarrow -\infty} \varphi_2(E) &= -\frac{\pi}{2} \Rightarrow \lim_{E \rightarrow -\infty} \varphi_3(E) = -\frac{3\pi}{2} \dots \Rightarrow \\ &\Rightarrow \lim_{E \rightarrow -\infty} \varphi_{N+1}(E) = -(2N - 1) \frac{\pi}{2}, \end{aligned} \quad (2.12)$$

$$\lim_{E \rightarrow +\infty} \varphi_2(E) = \frac{\pi}{2} \Rightarrow \lim_{E \rightarrow +\infty} \varphi_3(E) = \frac{\pi}{2} \dots \Rightarrow \lim_{E \rightarrow +\infty} \varphi_{N+1}(E) = \frac{\pi}{2} \quad (2.13)$$

By continuity and monotonicity of $\varphi_{N+1}(E)$, there exist exactly N different solutions of $\varphi_{N+1}(E) = k\pi$ for $E \in (-\infty, +\infty)$. \square

Proposition 5. *The ratio of derivatives $\frac{\varphi'_{n+1}(E)}{\varphi'_n(E)}$ is bounded above.*

Proof. By (2.10) :

$$\begin{aligned} \varphi'_{n+1}(E) &= \frac{d\varphi_{n+1}(E)}{dE} = \frac{1}{1 + (E - \varepsilon_n - \cot \varphi_n)^2} + \frac{\frac{\varphi'_n}{\sin^2 \varphi_n}}{1 + (E - \varepsilon_n - \cot \varphi_n)^2} \leq \\ &\leq 1 + \frac{\frac{\varphi'_n}{\sin^2 \varphi_n}}{1 + (E - \varepsilon_n - \cot \varphi_n)^2} = 1 + \frac{\varphi'_n}{1 - 2(E - \varepsilon_n) \sin \varphi_n \cos \varphi_n + (E - \varepsilon_n)^2 \sin^2 \varphi_n} \end{aligned} \quad (2.14)$$

It is left to find a lower bound for the denominator:

$$\begin{aligned} q &= 1 - 2(E - \varepsilon_n) \sin \varphi_n \cos \varphi_n + (E - \varepsilon_n)^2 \sin^2 \varphi_n = \\ &= 1 - (E - \varepsilon_n) \sin 2\varphi_n + (E - \varepsilon_n)^2 \sin^2 \varphi_n. \end{aligned} \quad (2.15)$$

For convenience, we define $x := E - \varepsilon_n$, then:

$$q(x, \varphi_n) = 1 - 2x \sin \varphi_n \cos \varphi_n + x^2 \sin^2 \varphi_n. \quad (2.16)$$

We are only interested in cases where $|x| = |E - \varepsilon_n| \leq W + 1$, because otherwise E is out of the spectrum interval, and cannot be in an interval between any pair of eigenstates. Furthermore: (2.15) is π -periodic in φ_n . Therefore, we are looking at a bound on the q , on the *compact* set (closed rectangle):

$$A = \{|x| \leq W + 1\} \times \{0 \leq \varphi_n \leq \pi\}. \quad (2.17)$$

Due to continuity of q in both x and φ_n , it has minimum in A . Therefore, to show that q is bounded away from zero, it is sufficient to prove that q is positive in A . (the formal continuity of q in arguments (x, φ_n) is obvious, and not to be confused with continuity of functions φ_n, φ'_n with respect to x or E).

Given the expression (2.16) with fixed φ_n , it can be evaluated as quadratic function in x with minimum value:

$$\min_x \{1 - 2x \sin \varphi_n \cos \varphi_n + x^2 \sin^2 \varphi_n\} = \sin^2 \varphi_n. \quad (2.18)$$

Therefore $q(x, \varphi_n)$ is positive for any $\varphi_n \neq k\pi$, but also positive for $\varphi_n = k\pi$, since $\varphi_n = k\pi \Rightarrow q = 1$. We hence have:

$$\eta(W) := \min_{|E - \varepsilon_n| \leq W + 1, \varphi_n} \{q\} > 0. \quad (2.19)$$

(Calculation of an analytic expression for η is given in appendix Appendix B. It demonstrates the dependence of the bound on W .) Substituting into the derivative inequality (2.14), we obtain a recursive inequality:

$$\varphi'_{n+1}(E) \leq 1 + \frac{\varphi'_n}{\eta}. \quad (2.20)$$

□

Remark 6. No separate argument is required for $\varphi_n = k\pi$, since limit (2.11) exists, and is a particular case of (2.14).

Proof. [Proof of theorem 1 for Dirichlet b.c.]

Evaluating the inequality (2.20) recursively, from $\varphi'_2(E) \leq 1$ to $\varphi'_{N+1}(E)$ (*recall* $\varphi_2(E) = \arctan(E - \varepsilon_1)$) one obtains:

$$\begin{aligned} \varphi'_{N+1}(E) &\leq \underbrace{\left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta} \left(\dots \left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta} \left(1 + \frac{\varphi'_2}{\eta}\right)\right)\right)\right)\right)\right)\right)}_{N-1} \\ &\leq \underbrace{\left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta} \left(\dots \left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta} \left(1 + \frac{1}{\eta}\right)\right)\right)\right)\right)\right)\right)}_{N-1} = \frac{\eta^{-N-1}}{\eta^{-1}-1} \end{aligned} \quad (2.21)$$

Since no degeneracy occurs (By proposition 4) and since eigenvalues satisfy (2.5), and the derivative $\varphi'_{N+1}(E)$ satisfies (2.21) for any $-1 \leq E \leq W + 1$, we have:

$$|E_i - E_j| \geq \frac{\pi(\eta^{-1}(W)-1)}{\eta^{-N}(W)-1} \text{ for any pair of eigenstates } E_i, E_j. \quad \square$$

2.3.2. *Proof for Neumann boundary conditions* Neumann b.c. imply:

$$u_1 = u_0 \neq 0 \Rightarrow \varphi_1(E) \equiv \frac{\pi}{4}. \quad (2.22)$$

The (2.5) is modified to be:

$$\varphi_{N+1} = (k + 1/4)\pi \Leftrightarrow u_{N+1} = u_N \Leftrightarrow E - \text{eigenvalue}. \quad (2.23)$$

The derivative $\varphi_2(E)' > 0$, and therefore proposition 3 holds. Limits in proposition 4 hold starting from $\varphi_2(E)$. Proposition 4 holds without modifications. Thus theorem 1 holds for Neumann b.c. as well.

2.3.3. *Simplicity of the spectrum for general boundary conditions*

Theorem 7. *Let H be defined as in theorem 1 with boundary conditions such that the normalization determines the value of the eigenfunction at two adjacent points. Then the spectrum of H is simple.*

Proof. Let \vec{v} and \vec{u} be eigenvectors of H with eigenvalue E_0 . Without loss of generality \vec{v} and \vec{u} , can be normalized so that $u_1 = v_1 = a$ and $u_0 = v_0 = b$. But from this point all the remaining elements of \vec{v} and \vec{u} can be determined using (2.4). Therefore $\vec{v} \equiv \vec{u}$. \square

2.4. *Remarks on level repulsion*

- (i) As expected, the limit on spacing vanishes when W approaches infinity. That is, when nearly infinite barriers are allowed (See (B.6), (B.7)).
- (ii) The typical sensitivity to the energy of the angle variable φ_{N+1} is exponentially large in N at the proximity of an eigenvalue (See appendix Appendix A)
- (iii) The result does not hold for periodic boundary conditions. For periodic boundary conditions, degeneracy might occur, and there is *no* lower bound on the energy spacing between non-degenerate states. For example consider $\varepsilon_j \equiv 0$. The eigenvalues

$$E_j = 2 \cos \frac{2\pi j}{N} \quad j = 1..N$$

are pair wise degenerate, except $j = N$ and $j = N/2$ (in case of even N).

- (iv) This work does not prove that the proposed limit is optimal. However, when considering exponential localization in disordered potentials, it appears that optimal bound on inter-level spacings is indeed exponential in chain length N .

3. Applications

3.1. *Bound on first order term in perturbation theory*

In this Section we will show some applications of Theorem 1. Our main interest will be in problems related to the still open question of whether there is localization for the Anderson model perturbed by a small nonlinearity. The numerical results so far are

inconclusive, see e.g. [15, 16, 17, 9, 18]. The only rigorous result that applies to the full nonlinear system, is the finite time result of [19]. In recent works [11, 12, 13], we developed a renormalized perturbation expansion for the nonlinear problem. The first order term, can now be controlled rigorously, as we will show now. The correction is given by the following term : (see (1.4) and (1.5)),

$$c_n^{(1)} = V_n^{000} \left(\frac{1 - e^{i(E_n - E_0)t}}{E_n - E_0} \right) \quad (3.1)$$

where,

$$V_n^{000} = \sum_y u_n(y) u_0^3(y), \quad (3.2)$$

We define,

$$\psi^{(1)}(x, t) = \sum_n c_n^{(1)}(t) u_n(x). \quad (3.3)$$

This is the correction to the wave function in first order perturbation theory [11] (in this section lattice sites are denoted by x and y , while eigenstates by n). Following [3] we are interested in bounding the fractional moments of $|\psi^{(1)}|$,

$$|\psi^{(1)}(x, t)|^s \leq \sum_n |c_n^{(1)}(t) u_n(x)|^s, \quad (3.4)$$

for $0 < s < 1$. Namely,

$$\begin{aligned} \langle |\psi^{(1)}(x, t)|^s \rangle &\leq \sum_n \langle |c_n^{(1)}(t) u_n(x)|^s \rangle \\ &\leq \sum_{y,n} \int d\mu(\omega) \left| \frac{u_n(x) u_n(y) u_0^3(y)}{E_n - E_0} \right|^s, \end{aligned} \quad (3.5)$$

where $d\mu(\omega)$ is the measure which is defined on the random potentials.

Lemma 8. *For any Hamiltonian in a finite box Λ of size N , with orthonormal eigenfunctions, $\sum_x u_n(x) u_m(x) = \delta_{n,m}$. For any x ,*

$$\exists n, \quad |u_n(x)| \geq N^{-1/2}. \quad (3.6)$$

Proof. Assume that $\exists x$ such that $\forall n, |u_n(x)| < N^{-1/2}$. Due to orthonormality of the eigenfunctions of a Hamiltonian,

$$1 = \sum_n |u_n(x)|^2 < N^{-1} \sum_n 1 = 1,$$

which is in a contradiction to the assumption. \square

Theorem 9. *For the one-dimensional Anderson model, and for $0 < s < \frac{1}{5}$,*

$$\langle |\psi^{(1)}(x, t)|^s \rangle \leq C_s N^{9/4 - s/2} e^{-\mu s |x|}, \quad (3.7)$$

where C_s is a constant which depends only on s .

Proof. Lets define a set of potentials with the help of the dyadic decomposition,

$$\mathcal{V}_n(m) = \{\omega \mid |E_n - E_0| \in I_m\}, \quad (3.8)$$

where,

$$I_m \equiv [2^{-m-1}, 2^{-m}]. \quad (3.9)$$

The denominators of (3.5) cannot be arbitrarily small by Theorem 1 (eigenvalue repulsion),

$$|E_n - E_0| \geq C e^{-bN}, \quad (3.10)$$

where C and b are constants. Therefore combining the decomposition (3.8) with (3.5) and (3.10) yields,

$$\left\langle |\psi^{(1)}(x, t)|^s \right\rangle \leq \sum_{m=0}^M 2^{s(m+1)} \sum_{y, n} d\mu(\omega) \chi(\mathcal{V}_n(m)) |u_n(x) u_n(y) u_0^3(y)|^s, \quad (3.11)$$

with,

$$\frac{1}{\ln 2} (bN - \ln C) \leq \bar{b}N \equiv M, \quad (3.12)$$

and $\chi(\mathcal{V}_m(n))$ is the characteristic function of the set of potentials, $\mathcal{V}_m(n)$. Applying the generalized Hölder inequality one finds,

$$\begin{aligned} \left\langle |\psi^{(1)}(x, t)|^s \right\rangle &\leq \sum_{n, y} \sum_{m=0}^M 2^{s(m+1)} \left(\int d\mu(\omega) \chi(\mathcal{V}_m(n))^{p_1} \right)^{1/p_1} \\ &\quad \times \left(\int d\mu(\omega) |u_n(x) u_n(y)|^{sp_2} \right)^{1/p_2} \\ &\quad \times \left(\int d\mu(\omega) |u_0(y) u_0(0) u_0^{-1}(0)|^{3sp_3} \right)^{1/p_3} \end{aligned} \quad (3.13)$$

with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. To estimate,

$$J_1 = \left(\int d\mu(\omega) \chi(\mathcal{V}_m(n))^{p_1} \right)^{1/p_1}, \quad (3.14)$$

we use the fact that $\chi^{p_1} = \chi$ and Minami estimate for the probability to find at least two eigenvalues in an interval I (see (1.6)) [14]

$$\Pr \left(\text{Tr} P_{H_\omega^\Lambda}^{(\Lambda)}(I) \geq 2 \right) \leq (\pi \|\rho\|_\infty I N)^2. \quad (3.15)$$

Since, we are not interested in a particular energy interval, we will cover the energy band with I^{-1} intervals of size $I = |I_m| = 2^{-m-1}$, which gives,

$$J_1 \leq (\pi \|\rho\|_\infty N)^{2/p_1} 2^{-(m+1)/p_1}. \quad (3.16)$$

To bound,

$$J_2 = \left(\int d\mu(\omega) |u_n(x) u_n(y)|^{sp_2} \right)^{1/p_2}, \quad (3.17)$$

and

$$J_3 = \left(\int d\mu(\omega) |u_0(y) u_0(0) u_0^{-1}(0)|^{3sp_3} \right)^{1/p_3}, \quad (3.18)$$

we choose, $1/p_2 = s$ and $1/p_3 = 3s$, which sets, $1/p_1 = 1 - 4s$ and $s < \frac{1}{4}$. Then, we proceed by combining Lemma 8 to bound, $|u_0(0)|^{-1}$, with the result of Aizenman [1],

$$\sum_n \left(\int d\mu(\omega) |u_n(x) u_n(y)| \right) \leq D e^{-\mu|x-y|}, \quad (3.19)$$

where $\mu > 0$, and $D > 0$ are some constants. This yields,

$$J_2 \leq D^s e^{-\mu s|x-y|}, \quad (3.20)$$

and

$$J_3 \leq N^{3s/2} D^{3s} e^{-3\mu s|y|}. \quad (3.21)$$

Plugging (3.16), (3.20) and (3.21) back into (3.13) gives,

$$\left\langle |\psi^{(1)}(x, t)|^s \right\rangle \leq D^{4s} (\pi \|\rho\|_\infty)^{2(1-4s)} N^{3-13s/2} \left(\sum_{m=0}^M 2^{(5s-1)(m+1)} \right) \sum_y e^{-\mu s|x-y|} e^{-3\mu s|y|}. \quad (3.22)$$

Setting $s < \frac{1}{5}$, and using the triangle inequality for the last sum, we get,

$$\sum_{m=0}^M 2^{(5s-1)(m+1)} < \frac{1}{1 - 2^{(5s-1)}}, \quad (3.23)$$

and,

$$\sum_y e^{-\mu s|x-y|} e^{-3\mu s|y|} \leq e^{-\mu s|x|} \sum_y e^{-2\mu s|y|} \leq e^{-\mu s|x|} \frac{1}{1 - e^{-2\mu s}}. \quad (3.24)$$

Therefore for $0 < s < \frac{1}{5}$,

$$\left\langle |\psi^{(1)}(x, t)|^s \right\rangle \leq C_{s,\delta} N^{3-13s/2} e^{-\mu s|x|}, \quad (3.25)$$

with,

$$C_{s,\delta} = \frac{D^{4s} (\pi \|\rho\|_\infty)^{2(1-4s)}}{(1 - 2^{(5s-1)}) (1 - e^{-2\mu s})}.$$

□

Corollary 10. For $\nu > 0$ and $0 < s < \frac{1}{5}$,

$$\Pr \left(|\psi^{(1)}(x, t)| \geq C_s^{1/s} N^{3/s-13/2} e^{-(\mu-\nu/s)|x|} \right) \leq e^{-\nu|x|}. \quad (3.26)$$

Proof. Using Chebychev inequality,

$$\Pr \left(\left| \psi^{(1)}(x, t) \right|^s \geq A \right) \leq A^{-1} C_s N^{3-13s/2} e^{-\mu s |x|}, \quad (3.27)$$

and choosing,

$$A = C_s N^{3-13s/2} e^{-(\mu-\nu)s|x|}, \quad (3.28)$$

gives,

$$\Pr \left(\left| \psi^{(1)}(x, t) \right|^s \geq C_s N^{3-13s/2} e^{-(\mu s - \nu)|x|} \right) \leq e^{-\nu|x|}, \quad (3.29)$$

or

$$\Pr \left(\left| \psi^{(1)}(x, t) \right| \geq C_s^{1/s} N^{3/s-13/2} e^{-(\mu-\nu/s)|x|} \right) \leq e^{-\nu|x|}. \quad (3.30)$$

□

3.2. Bound on the derivative of an eigenfunction

An important object in the study of the properties of eigenfunctions, is the sensitivity to a change of the potential at some point of an eigenfunction. We have by direct computation,

$$\frac{\partial u_0(x)}{\partial \varepsilon_y} = u_0(y) \sum_{n \neq 0} \frac{u_n(x) u_n(y)}{E_0 - E_n}. \quad (3.31)$$

The above analysis could be extended to obtain bounds on this derivative.

Theorem 11. *For a one-dimensional Anderson problem, and $0 < s < \frac{1}{3}$,*

$$E_s \equiv \left\langle \left| \frac{\partial u_0(x)}{\partial \varepsilon_y} \right|^s \right\rangle \leq K_s N^{3-7s/2} e^{-\mu s |x-y|} e^{-\mu s |y|}. \quad (3.32)$$

Proof. Proceeding in a similar manner to the previous subsection,

$$\begin{aligned} E_s &\leq \sum_{n \neq 0} \left\langle \frac{|u_0(y) u_n(x) u_n(y)|^s}{|E_0 - E_n|^s} \right\rangle \\ &\leq \sum_{n \neq 0} \sum_{m=0}^M 2^{s(m+1)} \int d\mu(\omega) \chi(\mathcal{V}_n(m)) |u_0(y) u_n(x) u_n(y)|^s. \end{aligned} \quad (3.33)$$

Now use the generalized Hölder inequality,

$$\begin{aligned} E_s &\leq \sum_{n \neq 0} \sum_{m=0}^M 2^{s(m+1)} \left(\int d\mu(\omega) \chi^{p_1}(\mathcal{V}_n(m)) \right)^{1/p_1} \left(\int d\mu(\omega) |u_n(x) u_n(y)|^{sp_2} \right)^{1/p_2} \\ &\quad \times \left(\int d\mu(\omega) |u_0(y) u_0(0) u_0(0)^{-1}|^{sp_3} \right)^{1/p_3}, \end{aligned} \quad (3.34)$$

with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. Setting $1/p_2 = 1/p_3 = s$, we have $1/p_1 = 1 - 2s$. Than using (3.16),

$$E_s \leq (\pi \|\rho\|_\infty N)^{2(1-2s)} \left(\sum_{m=0}^M 2^{(3s-1)(m+1)} \right) \times \sum_{n \neq 0} \left(\int d\mu(\omega) |u_n(x) u_n(y)| \right)^s \left(\int d\mu(\omega) |u_0(y) u_0(0) u_0(0)^{-1}| \right)^s. \quad (3.35)$$

Bounding the sum for $s < \frac{1}{3}$, utilizing the result of Aizenman (3.19) and Lemma 8 for the last term gives,

$$E_s \leq K_s N^{3-7s/2} e^{-\mu s|x-y|} e^{-\mu s|y|},$$

with

$$K_s = \frac{D^{2s} (\pi \|\rho\|_\infty)^{2(1-2s)}}{1 - 2^{1-3s}}.$$

□

3.3. Ordinary averages, $s = 1$

Note, that in the previous bounds the result of Theorem 1 were not used, since s was selected such that the sums 3.23 and 3.35 were convergent even for unbounded M . In the following we will calculate the bounds on the ordinary averages, namely for $s = 1$, and then interpolate between those two results.

Theorem 12. For the one-dimensional Anderson model on a box Λ of size N , and $\psi^{(1)}(x, t)$ defined in (1.4),

$$\langle |\psi^{(1)}(x, t)| \rangle \leq A N^{13/2}. \quad (3.36)$$

Proof. For the first order correction of the wavefunction we get (substituting $s = 1$ in (3.13)),

$$\begin{aligned} \langle |\psi^{(1)}(x, t)| \rangle &\leq \sum_{n,y} \sum_{m=0}^M 2^{(m+1)} \left(\int d\mu(\omega) \chi(\mathcal{V}_m(n))^{p_1} \right)^{1/p_1}, \\ &\times \left(\int d\mu(\omega) |u_n(x) u_n(y)|^{p_2} \right)^{1/p_2} \\ &\times \left(\int d\mu(\omega) |u_0(y) u_0(0) u_0^{-1}(0)|^{3p_3} \right)^{1/p_3} \end{aligned} \quad (3.37)$$

Using the bounds (3.16), (3.20) and (3.21) gives,

$$\begin{aligned} \langle |\psi^{(1)}(x, t)| \rangle &\leq D^{1/p_2+1/p_3} (\pi \|\rho\|_\infty)^{2/p_1} N^{5/2+2/p_1} \\ &\left(\sum_{m=0}^M 2^{(1-1/p_1)(m+1)} \right) \sum_y e^{-\mu|x-y|/p_2} e^{-\mu|y|/p_3}. \end{aligned} \quad (3.38)$$

Setting, $1/p_1 = 1 - \epsilon$ and $1/p_2 = 1/p_3 = \epsilon/2$ and using the triangle inequality for the last sum, yields

$$\langle |\psi^{(1)}(x, t)| \rangle \leq \frac{D^\epsilon (\pi \|\rho\|_\infty)^{2(1-\epsilon)}}{(1 - e^{-\mu\epsilon/2})(1 - 2^{-\epsilon})} N^{5/2+2(1-\epsilon)} 2^{M\epsilon}. \quad (3.39)$$

Since $M = \bar{b}N$ (3.12) we will set $\epsilon = 1/N$ to remove the exponential dependence on N , this gives,

$$\langle |\psi^{(1)}(x, t)| \rangle \leq \frac{D^{1/N} 2^{\bar{b}} (\pi \|\rho\|_\infty)^{2(1-1/N)}}{(1 - e^{-\mu/(2N)})(1 - 2^{-1/N})} N^{5/2+2(1-1/N)}, \quad (3.40)$$

or,

$$\langle |\psi^{(1)}(x, t)| \rangle \leq A N^{13/2}, \quad (3.41)$$

with

$$A = \frac{2^{\bar{b}+1} \pi^2 \|\rho\|_\infty^2 D}{\mu \ln 2}. \quad (3.42)$$

□

Theorem 13. For the one-dimensional Anderson model on a box Λ of size N ,

$$\left\langle \left| \frac{\partial u_0(x)}{\partial \varepsilon_y} \right| \right\rangle \leq B N^{11/2}, \quad (3.43)$$

Proof. Similarly, for the bound on the derivative, found from (3.34) by substituting $s = 1$,

$$\begin{aligned} E_1 &\leq \sum_{n \neq 0} \sum_{m=0}^M 2^{(m+1)} \left(\int d\mu(\omega) \chi^{p_1}(\mathcal{V}_n(m)) \right)^{1/p_1} \left(\int d\mu(\omega) |u_n(x) u_n(y)|^{p_2} \right)^{1/p_2} \\ &\quad \times \left(\int d\mu(\omega) |u_0(y) u_0(0) u_0(0)^{-1}|^{p_3} \right)^{1/p_3}, \end{aligned} \quad (3.44)$$

we get,

$$E_1 \leq D^{1/p_2+1/p_3} (\pi \|\rho\|_\infty)^{2/p_1} N^{5/2+2/p_1} \left(\sum_{m=0}^M 2^{(1-1/p_1)(m+1)} \right) e^{-\mu|x-y|/p_2} e^{-\mu|y|/p_3}, \quad (3.45)$$

setting as before $1/p_1 = 1 - \epsilon$, $1/p_2 = 1/p_3 = \epsilon/2$ gives,

$$E_1 \leq \frac{D^\epsilon (\pi \|\rho\|_\infty)^{2(1-\epsilon)}}{(1 - 2^{-\epsilon})} N^{9/2-2\epsilon} 2^{\epsilon M} e^{-\mu\epsilon|x-y|/2} e^{-\mu\epsilon|y|/2}. \quad (3.46)$$

Since $M = \bar{b}N$ (3.12) we will set $\epsilon = 1/N$ to remove the exponential dependence on N ,

$$E_1 \leq \frac{D^{1/N} 2^{\bar{b}} (\pi \|\rho\|_\infty)^{2(1-1/N)}}{(1 - 2^{-1/N})} N^{9/2-2/N} e^{-\mu|x-y|/(2N)} e^{-\mu|y|/(2N)}, \quad (3.47)$$

or

$$\left\langle \left| \frac{\partial u_0(x)}{\partial \varepsilon_y} \right| \right\rangle \leq B N^{11/2}, \quad (3.48)$$

with

$$B = D2^{\bar{b}} (\pi \|\rho\|_{\infty})^2. \quad (3.49)$$

□

Remark 14. The result above is not optimal, in fact it can be shown, using a different argument, that (3.43) can be improved to be linear in N . This will be shown elsewhere.

Corollary 15. For the one-dimensional Anderson model on a box Λ of size N , and $0 < s \leq 1$,

$$\left\langle \left| \psi^{(1)}(x, t) \right|^s \right\rangle \leq A_s N^{(11s+3)/2} e^{-2\mu(1-s)|x|/9}. \quad (3.50)$$

and

$$\left\langle \left| \frac{\partial u_0(x)}{\partial \varepsilon_y} \right|^s \right\rangle \leq B_s N^{5s+1} e^{-2\mu(1-s)|x-y|/5} e^{-2\mu(1-s)|y|/5}. \quad (3.51)$$

Proof. Since $f(s) \equiv \langle |\cdot|^s \rangle$ is a holomorphic and bounded function for $0 < s \leq 1$, we utilize Hadamard three-line interpolation theorem. For the first order correction to the wavefunction using Theorem 9 for $s = 2/11$ and Theorem 12, gives,

$$\left\langle \left| \psi^{(1)}(x, t) \right|^s \right\rangle \leq (C_{2/11} N^{95/44} e^{-2\mu|x|/11})^{\vartheta} (A N^{13/2})^{1-\vartheta}, \quad (3.52)$$

with,

$$\vartheta = \frac{11}{9} (1 - s). \quad (3.53)$$

Leading to,

$$\left\langle \left| \psi^{(1)}(x, t) \right|^s \right\rangle \leq A_s N^{(191s+43)/36} e^{-2\mu(1-s)|x|/9} \leq A_s N^{(11s+3)/2} e^{-2\mu(1-s)|x|/9}. \quad (3.54)$$

Similarly, for the derivative of the eigenfunction combining Theorem 11 for $s = 2/7$ and Theorem 13, gives

$$\left\langle \left| \frac{\partial u_0(x)}{\partial \varepsilon_y} \right|^s \right\rangle \leq (K_{2/7} N^2 e^{-2\mu|x-y|/7} e^{-2\mu|y|/7})^{\vartheta} (B N^{11/2})^{1-\vartheta},$$

with

$$\vartheta = \frac{7}{5} (1 - s). \quad (3.55)$$

Or,

$$\begin{aligned} \left\langle \left| \frac{\partial u_0(x)}{\partial \varepsilon_y} \right|^s \right\rangle &\leq B_s N^{(49s+6)/10} e^{-2\mu(1-s)|x-y|/5} e^{-2\mu(1-s)|y|/5} \\ &\leq B_s N^{5s+1} e^{-2\mu(1-s)|x-y|/5} e^{-2\mu(1-s)|y|/5}. \end{aligned} \quad (3.56)$$

□

Remark 16. This Corollary suggests exponential bounds for both the first order correction of the wavefunction and the derivative of the eigenfunction for the whole range $0 < s < 1$.

3.4. Time dependent bound

In this subsection we will eliminate the exponential dependence on the volume (see (3.39)) of the bound on the average first order correction to the wavefunction, $\psi^{(1)}$, by using the apriori bound

$$\left| \frac{1 - e^{i(E_n - E_0)t}}{E_n - E_0} \right| \leq t. \quad (3.57)$$

Theorem 17. For the one-dimensional Anderson model on a box Λ of size N , and $\epsilon_0(t) < \epsilon < 1$, such that $\epsilon_0(t) \rightarrow 0$ for $t \rightarrow \infty$, and $t \leq 2^{\bar{b}N}$ with \bar{b} given by (3.12).

$$\langle |\psi^{(1)}(x, t)| \rangle \leq K_\epsilon N^{5/2+2(1-\epsilon)/3} t^{(2+\epsilon)/3} \log_2 t e^{-\mu|x|/2}. \quad (3.58)$$

Proof. We start with

$$\begin{aligned} \langle |\psi^{(1)}(x, t)| \rangle &\leq D^{1/p_2+1/p_3} (\pi \|\rho\|_\infty)^{2/p_1} N^{5/2+2/p_1} \\ &\quad \times \left(\sum_{m=0}^M 2^{(1-1/p_1)(m+1)} \right) \sum_y e^{-\mu|x-y|/p_2} e^{-\mu|y|/p_3}. \end{aligned} \quad (3.59)$$

and set, $1/p_2 = 1/2 - 2\epsilon$, $1/p_3 = (1/2 - \epsilon)$ and $1/p_1 = 3\epsilon$, for $0 < \epsilon < 1/3$. If one is not intrested in the t dependence of the bound one calculates,

$$\langle |\psi^{(1)}(x, t)| \rangle \leq 2D^{1-3\epsilon} \frac{(\pi \|\rho\|_\infty)^{6\epsilon}}{1 - e^{-\epsilon}} N^{5/2+6\epsilon} \left(\sum_{m=0}^M 2^{(m+1)(1-3\epsilon)} \right) e^{-\mu|x|(1/2-2\epsilon)}. \quad (3.60)$$

To obtain the time-dependent bound we split the sum to two parts,

$$S \equiv \sum_{m=0}^M 2^{(1-3\epsilon)(m+1)} \leq \sum_{m=0}^{M_1} 2^{(1-3\epsilon)(m+1)} + t \sum_{m=M_1+1}^M 2^{-3\epsilon(m+1)}, \quad (3.61)$$

where we have used the fact that,

$$\left| \frac{1 - e^{i(E_n - E_0)t}}{E_n - E_0} \right| \leq \min(t, 2^m), \quad (3.62)$$

which defines, $M_1 = \log_2 t$. Therefore, for sufficiently large t

$$S \leq 2t^{(1-3\epsilon)} \log_2 t + t \frac{2^{-3\epsilon(M_1+1)}}{1 - 2^{-3\epsilon}} \leq 2t^{(1-3\epsilon)} \left(\log_2 t + \frac{1}{1 - 2^{-3\epsilon}} \right) \leq 3t^{(1-3\epsilon)} \log_2 t, \quad (3.63)$$

and

$$\langle |\psi^{(1)}(x, t)| \rangle \leq K_\epsilon N^{5/2+6\epsilon} t^{(1-3\epsilon)} (\log_2 t) e^{-\mu|x|(1/2-2\epsilon)},$$

with

$$K_\epsilon = 6D^{1-3\epsilon} \frac{(\pi \|\rho\|_\infty)^{6\epsilon}}{1 - e^{-\epsilon}},$$

and $t \leq 2^{\bar{b}N}$. □

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Appendix A. Relation Between φ_{N+1} and the Normalized Amplitude

Definition 18. we define $z_n(E) := \frac{u_n}{u_{n-1}}$.

Corollary 19. $z_n(E) = \tan(\varphi_n(E))$

Lemma 20. Let E_0 be an eigenvalue of H with corresponding normalized eigenvector $\{v_n\}_{n=1}^N$, then, $v_N^2 = (\frac{dz_{N+1}(E)}{dE}|_{E=E_0})^{-1}$.

Proof. By differentiability of φ_{N+1} , $\frac{dz_{N+1}(E)}{dE}|_{E=E_0}$ exists. By (2.4) and (2.2) it holds:

$$0 = z_{N+1}(E_0) = E_0 - \varepsilon_N - \frac{1}{z_N(E_0)}. \quad (\text{A.1})$$

(For Neumann b.c $z_{N+1}(E_0) = 1$, the rest of the argument still applies.)

Differentiating the LHS of (A.1) w.r.t. E and ε_N , one obtains:

$$\frac{\partial z_{N+1}}{\partial E} dE + \frac{\partial z_{N+1}}{\partial \varepsilon_N} d\varepsilon_N = 0. \quad (\text{A.2})$$

Differentiating the RHS of (A.1) one finds: $\frac{\partial z_{N+1}}{\partial \varepsilon_N} = -1$, therefore using $\frac{dE}{d\varepsilon_N}|_{E=E_0} = v_N^2$ (resulting from Feynman-Hellman theorem) one finds:

$$\frac{dz_{N+1}(E)}{dE}|_{E=E_0} = \frac{d\varepsilon_N}{dE}|_{E=E_0} = v_N^{-2}. \quad (\text{A.3})$$

□

Corollary 21. For a random potential with Anderson localization the derivative φ'_n , typically takes exponentially large (in N) values.

Appendix B. Expression for $\eta(W)$

The minimum of q (as in (2.16)) can be either at local extremum point or at the boundaries of A (defined by (2.17)); we will check both cases:

(i) local extrema in inner points:

$$\begin{cases} \frac{\partial q}{\partial x} = 0 \Rightarrow \sin \varphi_n \cos \varphi_n = x \sin^2 \varphi_n & \implies \cos \varphi_n = x \sin \varphi_n & (\varphi_n \neq 0, \pi) \\ \frac{\partial q}{\partial \varphi_n} = 0 \Rightarrow 2x \cos 2\varphi_n = x^2 \sin 2\varphi_n & \implies 2 \cos 2\varphi_n = x \sin 2\varphi_n & (\varphi_n \neq 0, \pi, x \neq 0) \end{cases} \quad (\text{B.1})$$

The case $x = 0$ is not interesting: if indeed extrema obtained for $x = 0$, then $q(x = 0, \varphi_n) \equiv 1$. Elsewhere (B.1) implies $\sin^2 \varphi = 0$ ($\varphi = k\pi$); consequently it is not an inner point of A . That is: *no* local extrema in A with $q(x, \varphi_n) \neq 1$.

(ii) Boundaries of A :

$$\varphi_n = 0, \pi \quad \Rightarrow \quad q = 1 \quad (\text{B.2})$$

$$x = W + 1 \quad \Rightarrow \quad \varphi_n^{extr} = \frac{1}{2} \arctan \frac{2}{(W+1)} \left(+\frac{\pi}{2} \right) \quad (\text{B.3})$$

$$x = -(W + 1) \quad \Rightarrow \quad \varphi_n^{extr} = \frac{-1}{2} \arctan \frac{2}{(W+1)} \left(+\frac{\pi}{2} \right) \quad (\text{B.4})$$

(with superscript extr standing for an extremum value on the appropriate boundary.)

Substituting (B.3) and (B.4) in (2.16) gives the following four options:

$$q^{extr} = \begin{cases} 1 - (W+1) \sin \left(\arctan \frac{2}{W+1} \right) + (W+1)^2 \sin^2 \left(\frac{1}{2} \arctan \frac{2}{W+1} \right) \\ 1 - (W+1) \sin \left(\arctan \frac{2}{W+1} \right) + (W+1)^2 \cos^2 \left(\frac{1}{2} \arctan \frac{2}{W+1} \right) \\ 1 - (W+1) \sin \left(\arctan \frac{2}{W+1} \right) + (W+1)^2 \sin^2 \left(\frac{1}{2} \arctan \frac{2}{W+1} \right) \\ 1 - (W+1) \sin \left(\arctan \frac{2}{W+1} \right) + (W+1)^2 \cos^2 \left(\frac{1}{2} \arctan \frac{2}{W+1} \right) \end{cases} \quad (\text{B.5})$$

By the arguments that lead to (2.19), all the entries of (B.5) are positive. The minimum of q is hence:

$$\begin{aligned} \eta(W) &= \min_{|E-\varepsilon_n| \leq W+1, \varphi_n} \{q\} \quad (\text{B.6}) \\ &= 1 - (W+1) \sin \left(\arctan \frac{2}{W+1} \right) + \\ &\quad + (W+1)^2 \min \left\{ \sin^2 \left(\frac{1}{2} \arctan \frac{2}{W+1} \right), \cos^2 \left(\frac{1}{2} \arctan \frac{2}{W+1} \right) \right\} \end{aligned}$$

Using (2.18) combined with (B.3), (B.4) one can obtain a simpler, yet weaker bound on η :

$$\eta(W) \geq \min \left\{ \sin^2 \left(\frac{1}{2} \arctan \frac{2}{(W+1)} \right), \cos^2 \left(\frac{1}{2} \arctan \frac{2}{(W+1)} \right) \right\} \quad (\text{B.7})$$

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